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AN INTRODUCTION TO GUMBEL, OR EXTREME-VALUE PROBABILITY PAPER

TRAINING MANUAL

U S ENVIRONMENTAL PROTECTION AGENCY

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AN INTRODUCTION TO GUMBEL, OR EXTREME-VALUE PROBABILITY PAPER

by

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The use of probability paper as a rough, ready, and rapid graphical hand tool in the analysis of frequency type data is an accepted technique, generally. Although this is the computer age, in some situations manual data analysis is justified.

Working with a small sample is one example, before the data cards could be punched, the manual analysis would be completed.

Graphical manual analysis would be practicable in the study of a new product, process, etc., where no prior knowledge is available. The aim is to uncover the underlying unknown distribution, which could entail much trial and error.

A third use of probability paper analysis is to obtain rough estimates of population parameters. As will be shown later, obtaining these is simple once the data is plotted and the regression line of best fit is drawn.

There are many probability papers commercially available that can be used for the purposes outlined above. Most of them are designed for continuous type data. They are normal, log-normal, Weibull, Gumbel or extreme value, log Gumbel or log extreme value, logistic, and reciprocal.

Two others, of a different type, are the binomial and Poisson. The latter is used for analyzing discrete data that are Poisson distributed. The binomial paper is useful in testing different types of hypotheses, and can be used also to handle other type problems too numerous to mention here. Of the papers listed, the binomial has the greatest utility.

This paper is another in a series designed to introduce graphical analysis, utilizing probability papers, to those who have not been exposed to this technique.

The first usual attempt is to plot the data on normal probability paper. If the plotted data fall in a straight line, the data are normal. If they are not normal, the analysis continues.

The graph on normal paper generally points to the next move. If the data plot as concave up (horizontal axis is percentage or probability), the data are positively skewed or have a long-right tail. The next attempt at linearization is to plot the data on log-normal probability paper. If the data plot concave up on log-normal, the data are more skewed than log-normal, and Weibull probability paper or log-extreme probability paper should be used in the next attempt. If the data when plotted on log-normal are concave down (or convex), the data are less skewed than log-normal, and Gumbel or extreme-value probability paper should be tried next.

If the plotts on normal paper are concave down, the curve is skewed left or has a long-left tail. Interchanging the order of the data will change them from skewed left to skewed right, and the interchanged data can be plotted on log-normal or extreme-value paper.

In this paper we give an introduction to Gumbel, extreme-value distribution type I, or double-exponential distribution. It has been used very successfully in many disciplines. Some of the applications found in reference 2 are (1) floods, (2) aeronautics, (3) breaking strength of materials, (4) life tables, (5) extention time for bacteria, (6) radioactivity, and (7) stock market. Some other applications found in the literature are (1) rainfall, (2) temperature, (3) crop yields, (4) crop-hail losses, and (5) design of waterway bridges.

Gumbel (1954) cites a specific example of the use of extreme-value paper in an analysis of the floods of the Colorado River at Black Canyon from 1878 - 1929. From the analysis, he concludes that the return period of a flood of 400,000 cubic feet per second (design flood for the dam at that point) is about 3,500 years. The probability is 2/3 that the design flood would occur between 1,100 and 11,000 years from 1929. This appears to be a case of overdesign.

The study of the statistical theory of extreme value has resulted in three distributions, which have been labeled types I, II, and III by Gumbel (1958).

The mathematical expression for the Gumbel type I cumulative distribution is

$$F(y) = e^{-e^{-y}}, \tag{1}$$

where y is any real number and is a reduced variate. It plays the same role as the reduced variate z does in normal theory. The transformation from the original variate x to the unit normal is given by

$$z = (x - \mu)\ \sigma$$

The analogous linear transformation for the double exponential is

$$y = \alpha\,(x - u), \tag{2}$$

where α is a scale parameter introduced for reduction. u is not the mean as with the normal, but the modal value for the distribution.

In equation (1), F(y) is the probability of getting a value of y or less. For example, if y = 4, the probability of getting y $\leq$ 4 is given by

$$F(4) = e^{-e^{-4}} = 0.982.$$

The graph of equation (1) is shown in Figure 1. The F (y) intercept is 0.368, and for large y the curve is asymptotic to the line F (y) = 1.

The Gumbel density function is obtained from equation (1) by differentiation, with the result given by

$$f(y) = e^{-y}\, e^{-e^{-y}} = e^{-y-e^{-y}}$$

This graph, also plotted in Figure 1, has a long-right tail as a prominent feature.

Some values for a few characteristic measures of the reduced variate are:

$$\text{mode } (\tilde{y}) = 0$$

$$\text{mean } (\bar{y}) = \gamma \text{ (Euler's Constant)} \approx 0.577$$

$$\text{median } (\breve{y}) = 0.3665$$

$$\text{standard deviation} = \sigma_y = 1.28255$$

$$\text{skewness} = \beta_1 = 1.29857$$

$$\text{kurtosis} = \beta_2 = 27/5.$$

By way of comparison, $\beta_1 = 0$ for normal and $\beta_2 = 3$. One notes that there is a marked difference in the value of these two parameters.

Some values for key characteristic measures of the original variate are:

$$\text{mode } (\tilde{x}) = u$$

$$\text{mean } (\bar{x}) = u + 0.57722/\alpha$$

$$\text{median } (\breve{x}) = u + 0.3665/\alpha$$

$$\text{standard deviation} = \sigma_x = \sigma_y/\alpha = 1.28255/\alpha.$$

The construction of the probability paper for the type I distribution is accomplished as follows.

Draw two parallel horizontal lines. Label the bottom one "reduced variate," and scale off with a uniform measure (see Figure 2). Since

$$e^{-e^{-0}} = 0.36787,$$

the vertical projection of 0 on the reduced scale to the second parallel line is 0.36787.

In a similar manner, 1 is projected to 0.6922 since

$$e^{-e^{-1}} = 0.6922.$$

The vertical projection of 1 on the reduced scale to the second parallel is 0.6922. These projections are facilitated by using F (y) in Figure 1.

Thus, for every value on the reduced variate scale, there is a value on the probability scale. The values on the probability scale are so chosen that plotting on this scale is easy.

Once the probability scale has been chosen for a suitable range of values, any scaling can be used for the vertical axis. In Figure 3, the vertical axis is arithmetic, and in Figure 4, it is logarithmic.

In Figure 3 an arithmetic scale has been used on both axes for the original and reduced variate. From equation (2), it is seen that the relationship between y and x is linear in x and y; therefore, plotting y versus x will result in a straight line (see Figure 2).

In sampling, α and u are unknown and, hence, y is unknown. In place of y, one uses its probability scale. Its probability is a function of its ordered value and sample size only. The value is determined by the formula

$$\text{plotting position (as a \%)} = \frac{100\ i}{n + 1}.$$

Here i = 1 for the smallest ordered value, and i = n for the largest ordered value.

In Figures 3 and 4, the top horizontal scale is labeled "return period." Formally, it is defined as follows:

$$\text{return period} = \frac{1}{1 - \text{probability}}.$$

As stated by Gumbel, "it (return period) is the number of observations such that, on the average, there is one observation equaling or exceeding x."

Since the probability for the median or less is 1/2, its return period is 2.

$$\text{return period} = \frac{1}{1 - 1/2}\ 2.$$

Given two observations, on the average, there is one observation equaling or exceeding the median.

Another use which seems to justify the name is when observations are made over constant time intervals. In these cases, return period is related to time. In Figure 3, some fictional air pollution data were plotted, and the regression line of best fit was drawn. Each point represents the maximum value of SO_2 measured in a large American city during a 1-year period. Extrapolation of this or any data is hazardous, but is often a necessary and reasonable request. In this case, the problem can be stated in two ways. The first is as follows. How long will it be before SO_2 reaches 40 ppm?

In Figure 3, we draw a horizontal line from 40 ppm on the vertical scale until it reaches the line of best fit. This point is projected vertical upward to the return-period scale. The value there is 25. This means on the average, within 25 years, we expect to reach a value of 40 ppm. Precisely which year it will be, is lost. We are not treating these data as a time series, but as a frequency distribution.

The phrase "on the average" should be carefully noted and interpreted as repeated experiments of the time to reach 40 ppm would average 25 years. The elapsed time, therefore, could be as short as 1 year or occur on successive years. For others the gap could be much longer than 25 years.

A second use of the return period for extrapolation could be determining how high SO_2 levels will go during the next 50 years. In Figure 3, we start at return period 50, project vertical to the line of best fit, and read the corresponding ppm from the vertical scale. Some time within the next 50 years, on the average, one expects an SO_2 level of about 43 ppm.

We next consider plotting data on Gumbel paper. Two cases are considered.

First, the number of data points is small enough to hand plot values obtained in Table 1. Table 1 gives the plotting position for each value in the sample from size 2 to 50. These numbers represent the values recommended by Gumbel (1958).

Let the n original variates or measurements be $X_1, X_2, \ldots X_n$. Order them from smallest to largest. This is written notationally as

$$X_{(1)}, X_{(2)}, \ldots X_{(n)}$$

where $X_{(1)}$ is the smallest value and $X_{(n)}$ is the largest. The plotting positions for the ranked data are arranged in columns in Table 1. An example of their use is given in Table 2 and plotted in Figure 5. A straight line represents the data; hence, the distribution is Gumbel type I.

For rough and ready preliminary investigation and sample sizes greater than 50, the following technique works well and can be done easily by hand. If there are from 51 to 100 data points, plot every other one. If there are from 101 to 150, plot every third one, etc. Plotting more than 50 points for a preliminary investigation appears to be too much time wasted. To some, 50 is too high and not more than x (the reader should supply his own value for x) should be plotted.

Personally, if I had 134 points, then the plotting position for the third ordered value, or first point plotted, is:

$$\text{plotting position } (i = 3;\ n = 134) = \frac{100\,(3)}{135} = 2.222.$$

Since the plotted points are all equally spaced, adding 2.222 to the first plotting position gives the plotting position for the 6th ordered value. This repeated addition can be done easily by desk calculator or by hand.

After the 44th point (or the 132nd ordered value) is plotted, both ends of the distribution can be plotted by calculating the individual plotting positions. In this example, plot 1st, 2nd, 133rd, and 134th ordered values by hand. This example is illustrated in Table 3.

Generally before plotting end values, one can decide subjectively whether data are approximately linear or not, hence, those points can be omitted.

Another labor saver that can be used is the rounding of the plotting positions. Consider the problem of plotting 160 points. Every 4th value would be plotted, and the first plotting position is

$$\text{plotting position } (i = 4,\ n = 160) = \frac{100\,(4)}{161} = 2.484472.$$

This is very close to 2.50, so close that one cannot plot each separately. The plotted points are put 2.50 apart up to 50.0. Next, we work from the other end to the middle by fixing the end point and subtracting until the middle is reached. The largest errors are near 50.0 where small errors (50.0 versus correct 49.69) cause essentially no effect. In this case, nothing is added or lost by omitting the 84th ordered value. See Table 4, column 2, for these values.

The reader might object to starting at both ends and working toward the middle since more work is involved by doing it this way. Some answers to this objection are as follows.

In this particular example, the difference between the true value and the approximation was very small. Hence, the difference in all plotting positions between the two values (correct versus approximate) was essentially the same whether one started at both ends and worked toward the middle, or started at one end and worked toward the other. In some problems where the approximation requires much rounding, the difference between the two techniques is very marked.

Comparison of columns 2 and 3 of Table 4 exhibits another objection to starting at one end only. The point at the 160th ordered value cannot be plotted since the 100 percent point is absent on probability paper. If the rounding resulted in values greater than 100 percent, the result would be more incorrect.

Another point to be noted is that when one works from end points toward the middle, there are no errors at the tails and the largest error is in the middle. This means that an error on tails represents a larger horizontal distance or incorrect displacement. The same size error near the 50 percent point would be a small horizontal displacement. This is the nature and construction of the probability scale.

The regression line is a function of the horizontal errors, and these errors should be minimized since the parameters are graphically estimated from the line. The line would be most affected by larger errors at one end only.

There are no hard and fast rules except speed, efficiency, and using the minimum of data that you need to represent the true situation.

Sometimes one has to plot a frequency distribution where the individual values are lost. One solution is to plot cumulative frequency versus end of the class interval. This is shown in Table 5, where the last interval cannot be plotted. One plots 4 at 5 percent, 9 at 40 percent, etc. The last interval, of course, is lost. Plotting of individual values is preferred.

After the data have been plotted and are subjectively determined as linear, the regression line of best fit is drawn.

The well-known analytic technique of minimizing squared deviations is normally used. This method cannot be applied when one subjectively draws the line. Instead, the line must be drawn so that the sums of the absolute values of the deviations above and below the line are equal.

In practice, this is more difficult to do than to describe. Ferrell (1958) suggests the following excellent iterative method. Use a transparent straight line (like the edge of a draftsman triangle) to obtain the initial line of what looks like a best fit. This is the first iteration, and the line should not be drawn. Fix a pivot point on this straight edge in the lower left-hand corner of the graph near the smallest ordinal value. Rotate the straight edge about the initial pivot. The pivoted transparent straight edge is rotated to obtain a best fit for the right half of the data, i.e., the data on the right side of the graph paper only. This is the second iteration, and this line also should not be drawn. While keeping the straight edge fixed after the second line of best fit has been determined, shift the pivot point to the upper right-hand corner nearest the largest value. Rotate the straight edge around the second pivot point to obtain a line of best fit for the left half of the data only. This is the third iteration, and this line is drawn. The fourth iteration consists of shifting the pivot from right to left. Again the straight edge is rotated about the third pivot point, minimizing the distance on the right half of the data again. This line is generally coincident with the previous one. If not, it is drawn, and the previous one is discarded. Other iterations are made until two coincident lines in a row are obtained. Generally, three or four iterations, at most, are needed, since no further adjustment is made on the drawn straight line. The big advantage of this method is that one cuts in half the data points that are being minimized, by eye, with respect to deviations.

At this point, the goodness of fit is accepted subjectively, or a test for linearity is made. One nonparametric test appeals to run theory. The method is described by Crow, et al. (1960). He used it in a different situation, but it applies here also.

The graphical estimate of the parameters, after the regression line of best fit has been drawn by eye or subjectively, is quite simple.

Solving for x from equation (2), we get

$$x = u + \frac{y}{\alpha} . \qquad (3)$$

If y = 0, this equation reduces to

$$x = u.$$

In Figure 3 at y = 0 (0.360 on the probability scale), one reads that

$$26.4 = u.$$

If one substitutes $y = 1$ in equation 3, the result is

$$x = u + \frac{1}{\alpha}$$

Since u is known and x for $y = 1$ can be read from the regression line, the only unknown is $1/\alpha$. From Figure 3 for $y = 1$ (or probability = 0.692), we obtain

$$30.6 - 26.4 = \frac{1}{\alpha}$$

$$4.2 = \frac{1}{\alpha}$$

These, of course, are graphical estimates. Analytical estimates can be obtained by using the work sheets on pages 50 and 51 with Table 3.3 on page 31 of Gumbel (1954).

Following the work sheets, one not only gets the regression line of best fit but confidence bands for the regression line, as well as expected values for selected return periods.

There are cases when the original variate is not a Gumbel distribution, but some transformation will make it so. A log transformation is accomplished by plotting the original data on the graph paper (shown in Figure 4), and then working with the transformed data.

If the data do not plot as a straight line, then the shape of the distribution is still to be determined. Figures 6 to 12 give the results of plotting on double exponential when the underlying distributions is not double exponential. A conclusion that can be drawn from these graphs is that a finite distribution terminates at both ends in a horizontal line.

At times, data are recorded incorrectly, either being too large or small. The results of graphing data with the presence of outliers is shown in Figure 13. Curves A and C have outliers at one end, and in B they are present at both ends.

Another type of error is a truncated distribution in which data above or below a threshold value are lost or not recorded. This result is shown in Figure 14 where A and C again depict a loss at one end and curve B shows a lose at both ends. Misclassified data behave in the same way and would give the same sort of picture. In curve A, the misclassification is at the one end, that is, data are recorded at a higher value than they should be. Analogous interpretations should be given for curves B and C in Figure 14 for misclassified data.

There are times when the data plotted are actually the sum of two Gumbel distributions. Such a result is shown in Figure 15. Curve A is the result of adding two distributions whose ratio of sample size is roughly 2 to 1 as shown by the length of the two straight line segments. Each of these could be used to estimate the respective parameters. In curve B, the contribution of each distribution to the sum distribution is about equal.

Table 1. **PLOTTING POSITIONS FOR GUMBEL PROBABILITY PAPER**

Sample Size

Ordinal No.	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	Ordinal No.
1	33.3	25.0	20.0	16.7	14.3	12.5	11.1	10.0	9.1	8.3	7.7	7.1	6.7	6.3	5.9	5.6	5.3	5.0	4.8	4.5	4.3	4.2	4.0	3.8	3.7	3.6	3.4	3.3	3.2	1
2	66.7	50.0	40.0	33.3	28.6	25.0	22.2	20.0	18.2	16.7	15.4	14.3	13.3	12.5	11.8	11.1	10.5	10.0	9.5	9.1	8.7	8.3	8.0	7.7	7.4	7.1	6.9	6.7	6.5	2
3		75.0	60.0	50.0	42.9	37.5	33.3	30.0	27.3	25.0	23.1	21.4	20.0	18.8	17.6	16.7	15.8	15.0	14.3	13.6	13.0	12.5	12.0	11.5	11.1	10.7	10.3	10.0	9.7	3
4			80.0	66.7	57.1	50.0	44.4	40.0	36.4	33.3	30.8	28.6	26.7	25.0	23.5	22.2	21.1	20.0	19.0	18.2	17.4	16.7	16.0	15.4	14.8	14.3	13.8	13.3	12.9	4
5				83.3	71.4	62.5	55.6	50.0	45.5	41.7	38.5	35.7	33.3	31.3	29.4	27.8	26.3	25.0	23.8	22.7	21.7	20.8	20.0	19.2	18.5	17.9	17.2	16.7	16.1	5
6					85.7	75.0	66.7	60.0	54.5	50.0	46.2	42.9	40.0	37.5	35.3	33.3	31.6	30.0	28.6	27.3	26.1	25.0	24.0	23.1	22.2	21.4	20.7	20.0	19.4	6
7						87.5	77.8	70.0	63.6	58.3	53.8	50.0	46.7	43.8	41.2	38.9	36.8	35.0	33.3	31.8	30.4	29.2	28.0	26.9	25.9	25.0	24.1	23.3	22.6	7
8							88.9	80.0	72.7	66.7	61.5	57.1	53.3	50.0	47.1	44.4	42.1	40.0	38.1	36.4	34.8	33.3	32.0	30.8	29.6	28.6	27.6	26.7	25.8	8
9								90.0	81.8	75.0	69.2	64.3	60.0	56.3	52.9	50.0	47.4	45.0	42.9	40.9	39.1	37.5	36.0	34.6	33.3	32.1	31.0	30.0	29.0	9
10									90.9	83.3	76.9	71.4	66.7	62.5	58.8	55.6	52.6	50.0	47.6	45.5	43.5	41.7	40.0	38.5	37.0	35.7	34.5	33.3	32.3	10
11										91.7	84.6	78.6	73.3	68.8	64.7	61.1	57.9	55.0	52.4	50.0	47.8	45.8	44.0	42.3	40.7	39.3	37.9	36.7	35.5	11
12											92.3	85.7	80.0	75.0	70.6	66.7	63.2	60.0	57.1	54.5	52.2	50.0	48.0	46.2	44.4	42.9	41.4	40.0	38.7	12
13												92.9	86.7	81.3	76.5	72.2	68.4	65.0	61.9	59.1	56.5	54.2	52.0	50.0	48.1	46.4	44.8	43.3	41.9	13
14													93.3	87.5	82.4	77.8	73.7	70.0	66.7	63.6	60.9	58.3	56.0	53.8	51.9	50.0	48.3	46.7	45.2	14
15														93.8	88.2	83.3	78.9	75.0	71.4	68.2	65.2	62.5	60.0	57.7	55.6	53.6	51.7	50.0	48.4	15
16															94.1	88.9	84.2	80.0	76.2	72.7	69.6	66.7	64.0	61.5	59.3	57.1	55.2	53.3	51.6	16
17																94.4	89.5	85.0	81.0	77.3	73.9	70.8	68.0	65.4	63.0	60.7	58.6	56.7	54.8	17
18																	94.7	90.0	85.7	81.8	78.3	75.0	72.0	69.2	66.7	64.3	62.1	60.0	58.1	18
19																		95.0	90.5	86.4	82.6	79.2	76.0	73.1	70.4	67.9	65.5	63.3	61.3	19
20																			95.2	90.9	87.0	83.3	80.0	76.9	74.1	71.4	69.0	66.7	64.5	20
21																				95.5	91.3	87.5	84.0	80.8	77.8	75.0	72.4	70.0	67.7	21
22																					95.7	91.7	88.0	84.6	81.5	78.6	75.9	73.3	71.0	22
23																						95.8	92.0	88.5	85.2	82.1	79.3	76.7	74.2	23
24																							96.0	92.3	88.9	85.7	82.8	80.0	77.4	24
25																								96.2	92.6	89.3	86.2	83.3	80.6	25
26																									96.3	92.9	89.7	86.7	83.9	26
27																										96.4	93.1	90.0	87.1	27
28																											96.6	93.3	90.3	28
29																												96.7	93.5	29
30																													96.8	30

References:
(1) Probability Tables for the Analysis of Extreme-Value Data, by Gumbel, E. J., Natl. Bureau of Stand., Appl. Math. Ser. No. 22, '53
(2) Statistical Theory of Extreme Values and Some Practical Applications, by Gumbel, E.J., Natl. Bureau of Stand., Appl.Math.Ser.No.33, '54
(3) Statistics of Extremes, by Gumbel, E. J., Columbia University Press, New York, '58

Table 2. EXAMPLE OF PLOTTING ON GUMBEL PAPER

Sample data	
Ordered original variate	Plotting position from Table 1, %
7.6	10
8.1	20
10.2	30
11.6	40
13.4	50
13.7	60
17.0	70
17.4	80
22.8	90

Table 3. APPROXIMATE PLOTTING OF A SAMPLE OF SIZE 134

Ordered value		Plotting position as a percent
1		$\frac{100\ (1)}{135} = 0.74$[a]
2		$\frac{100\ (2)}{135} = 1.48$[a]
3		$2.222 = \frac{100\ (3)}{135}$
6	By addition ↓	4.444
9		6.666
.		
.		
.		
129		95.546
132		97.768
133		$\frac{100\ (133)}{135} = 98.52$[a]
134		$\frac{100\ (134)}{135} = 99.26$[a]

[a]Calculated separately to plot end values

Table 4. APPROXIMATING PLOTTING OF A SAMPLE OF SIZE 160

Ordered value		Plotting position as a percent working from both ends to the middle		Plotting position as a percent working from one end only
4	By addition	$\frac{100\,(4)}{161} = 2.484472 \approx 2.5$	By addition	2.5
8	↓	5.0	↓	5.0
.				
.				
.				
80	↓	50.0	↓	50.0
84		49.38	↓	52.5
88	↑	51.88	↓	55.0
.				
.				
.				
156	↑	96.88	↓	97.5
160	By subtraction	$\frac{100\,(160)}{161} = 99.38$		100.00

Table 5. PLOTTING A FREQUENCY DISTRIBUTION

Class interval	Frequency	Percent frequency	Cumulative frequency
1 - 4	10	5	5
5 - 9	70	35	40
10 - 14	84	42	82
15 - 19	32	16	98
20 - 24	4	2	100

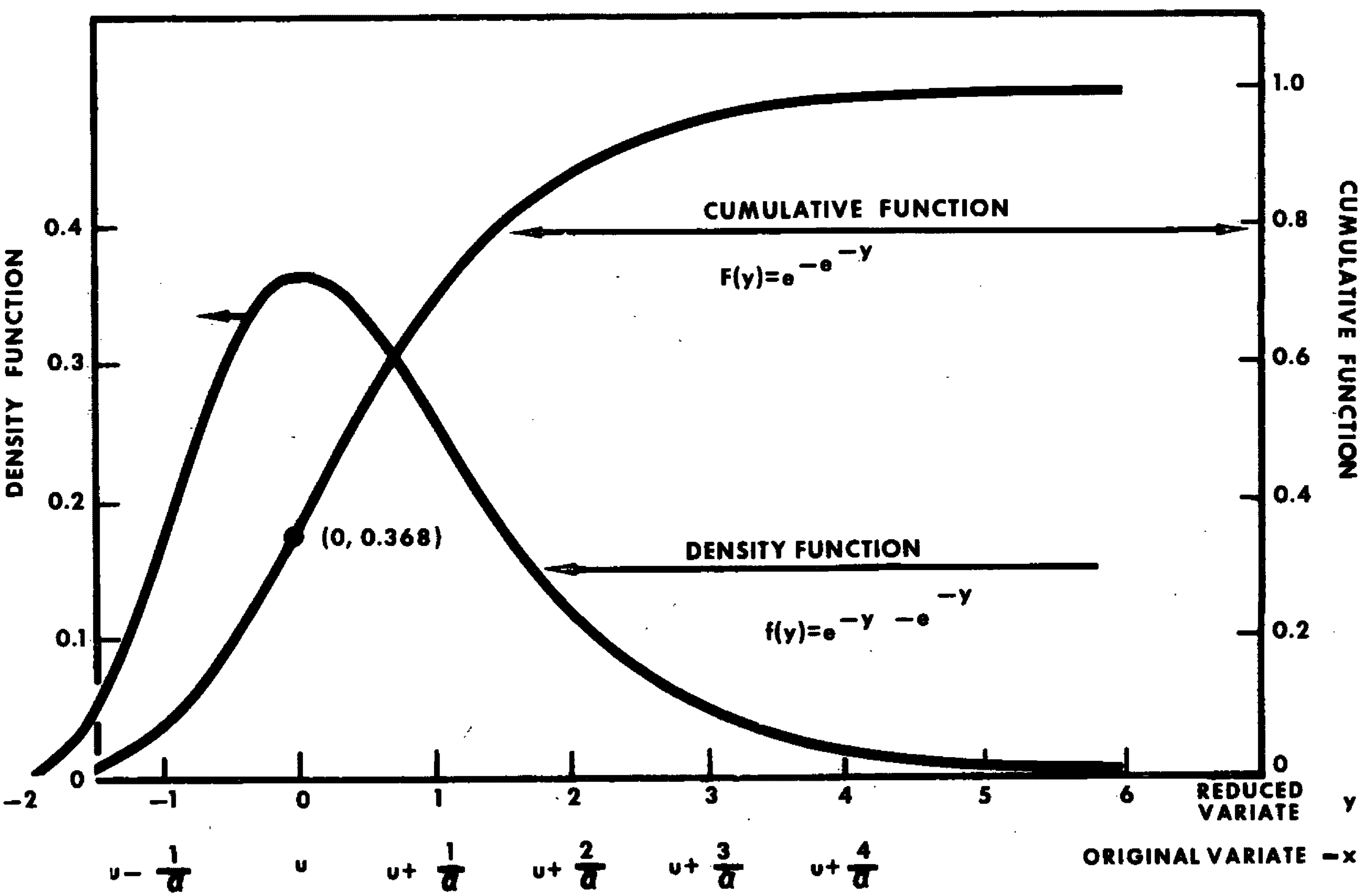

Figure 1. EXTREME VALUE DENSITY AND CUMULATIVE FUNCTIONS

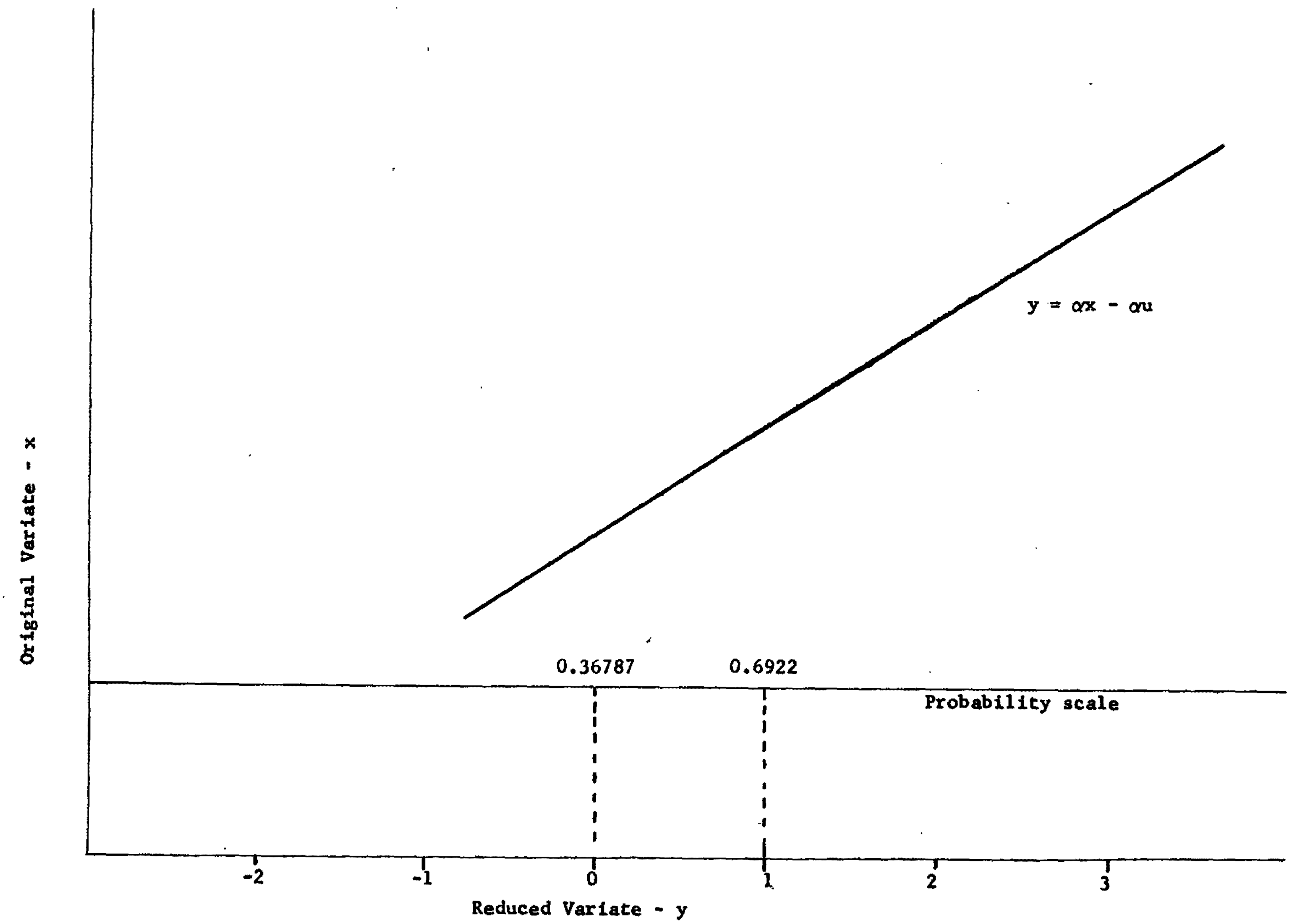

Figure 2 Construction of Gumbel probability paper.

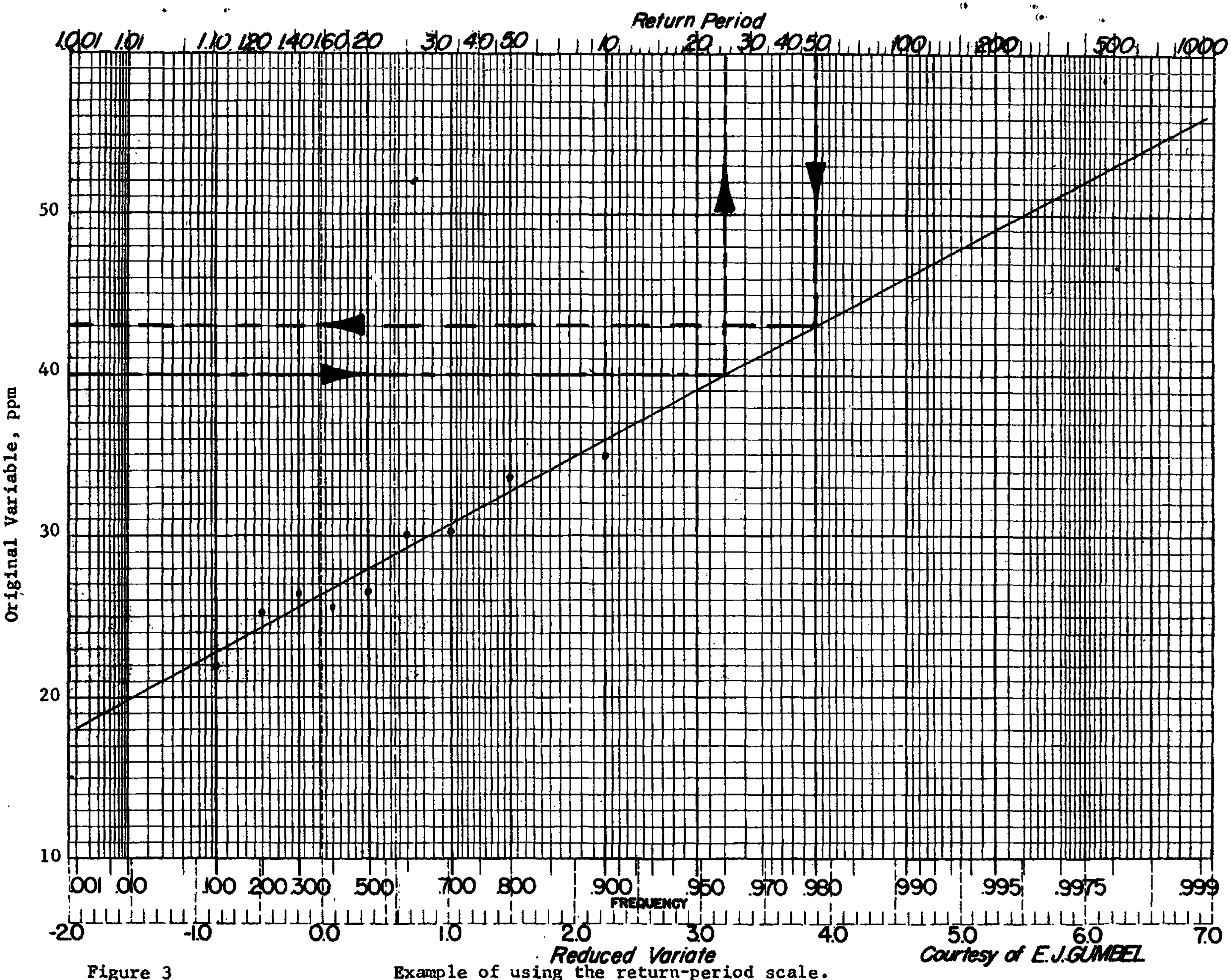

Figure 3 Example of using the return-period scale.

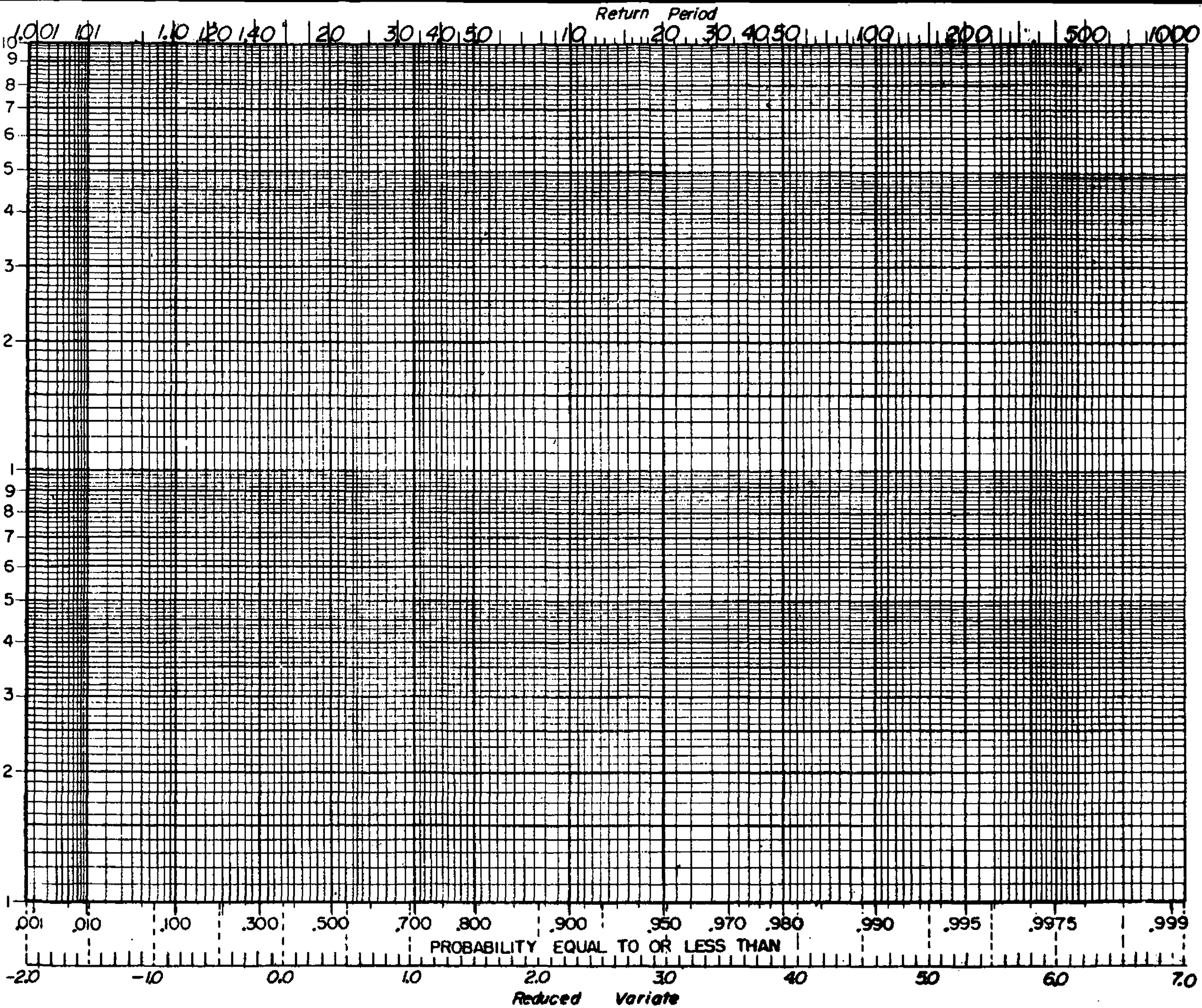

Fig. 4 Log-Gumbel probability paper.

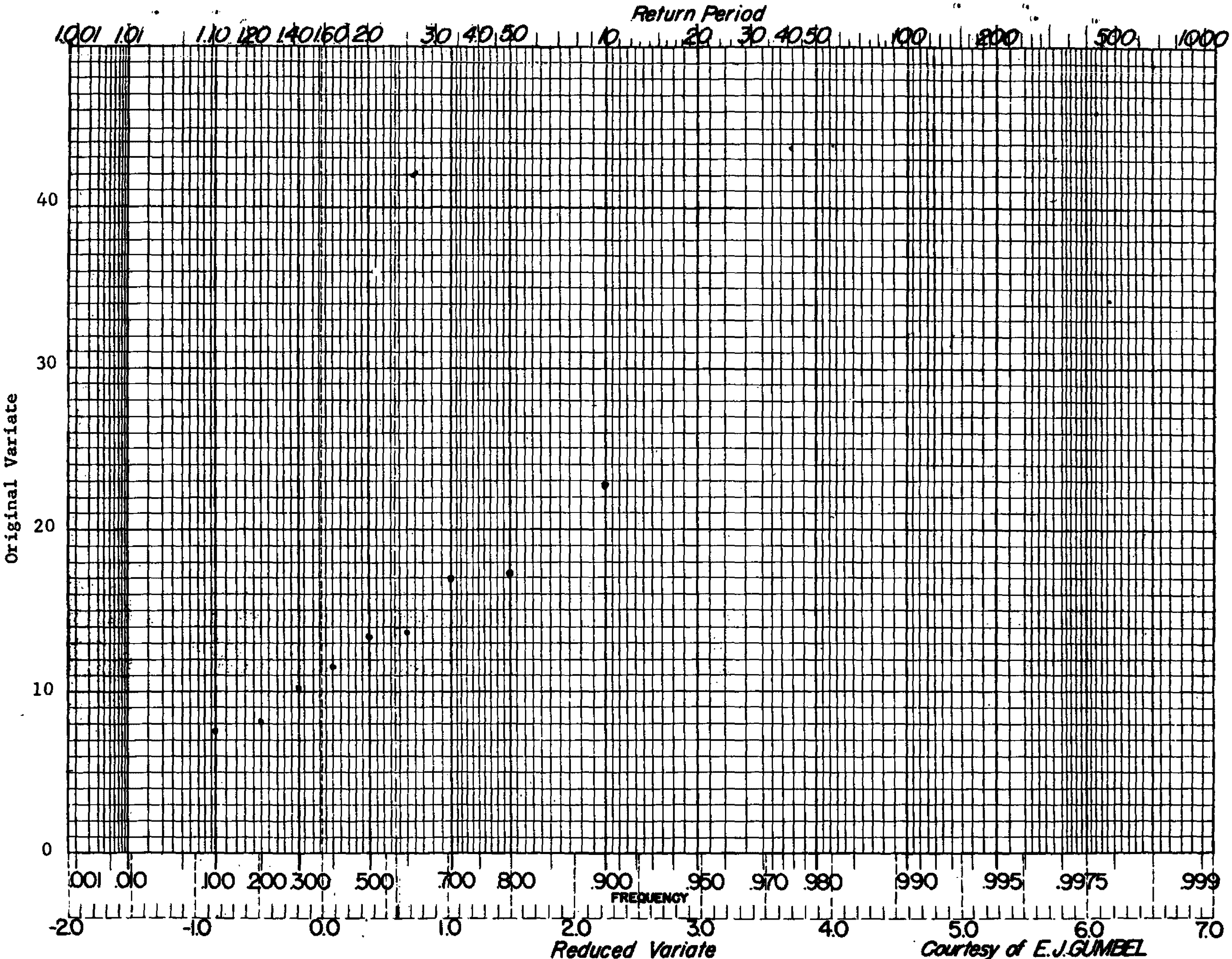

Figure 5 Example of plotting on Gumbel paper.

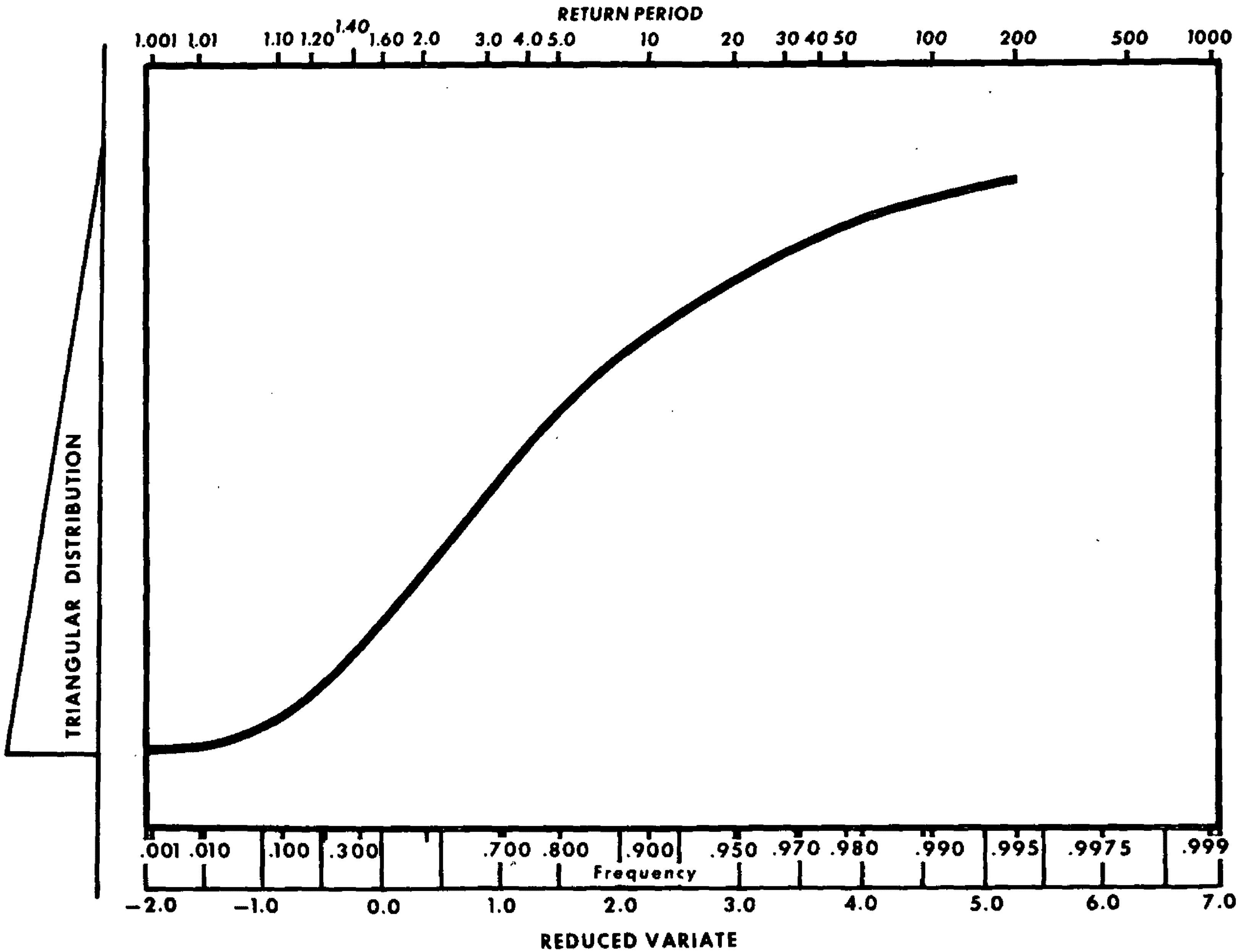

Figure 6. TRIANGULAR DISTRIBUTION ON GUMBEL PAPER

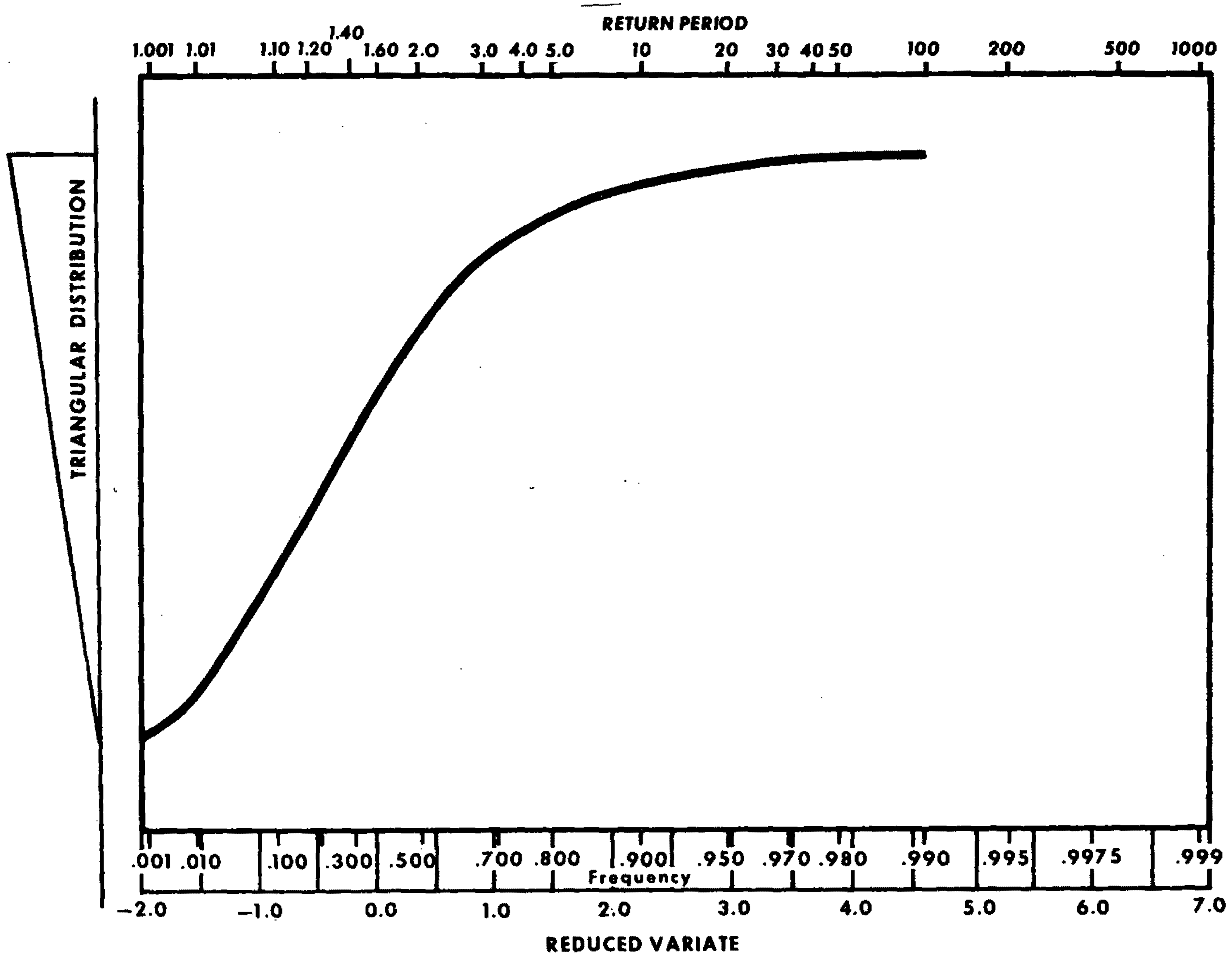

Figure 7. TRIANGULAR DISTRIBUTION ON GUMBEL PAPER

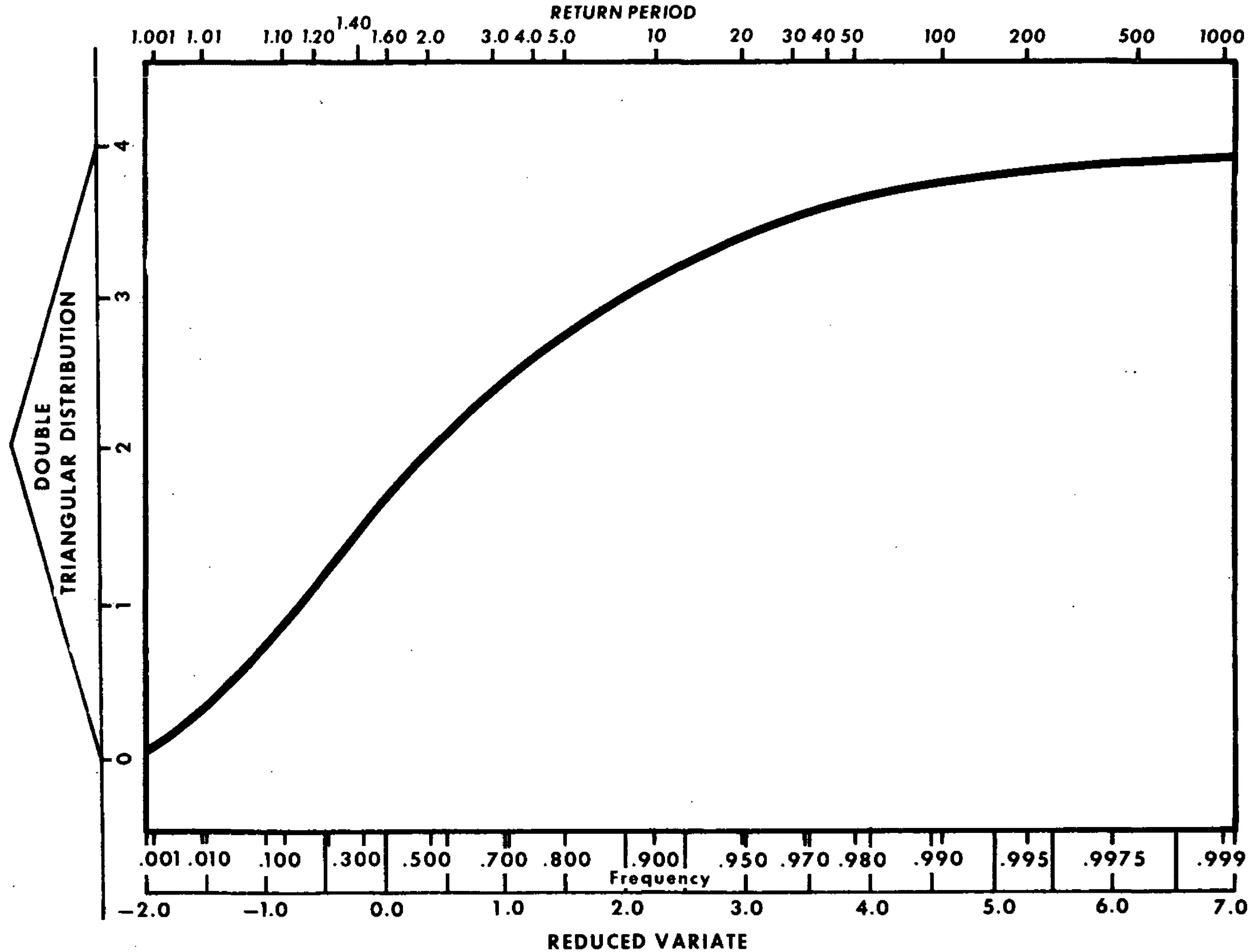

Figure 8. DOUBLE TRIANGULAR DISTRIBUTION ON GUMBEL PAPER

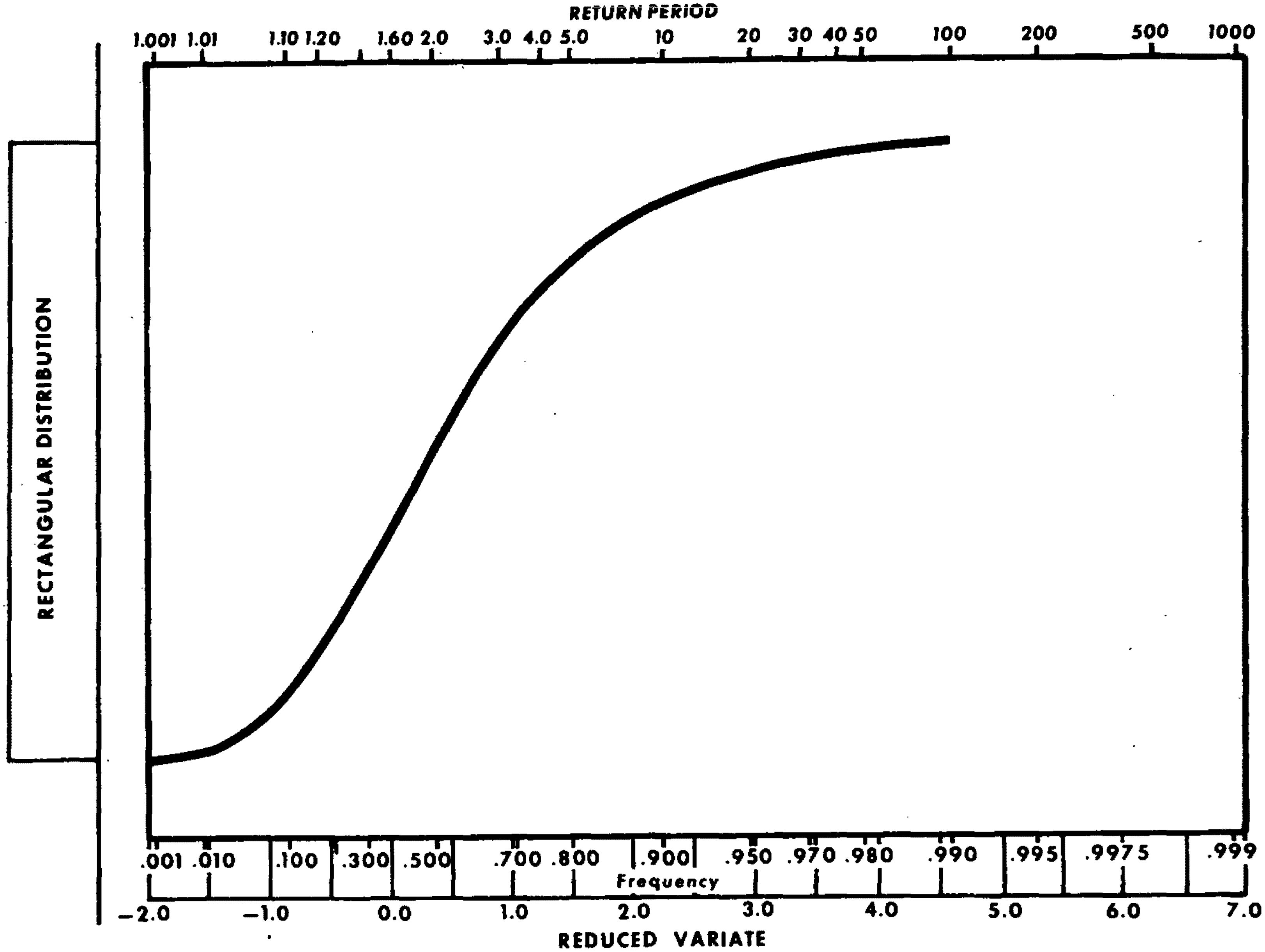

Figure 9. RECTANGULAR DISTRIBUTION ON GUMBEL PAPER

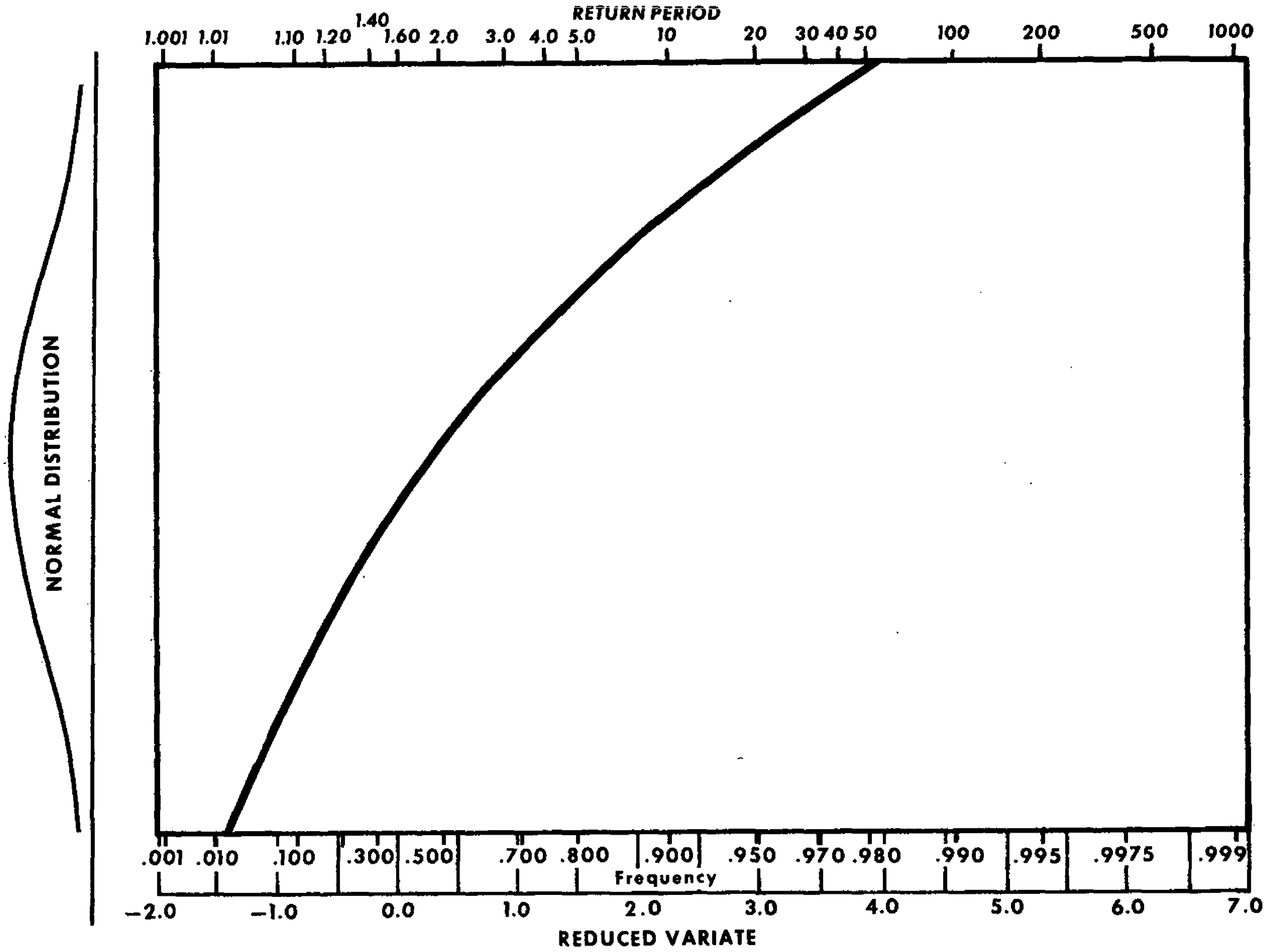

Figure 10. NORMAL DISTRIBUTION ON GUMBEL PAPER

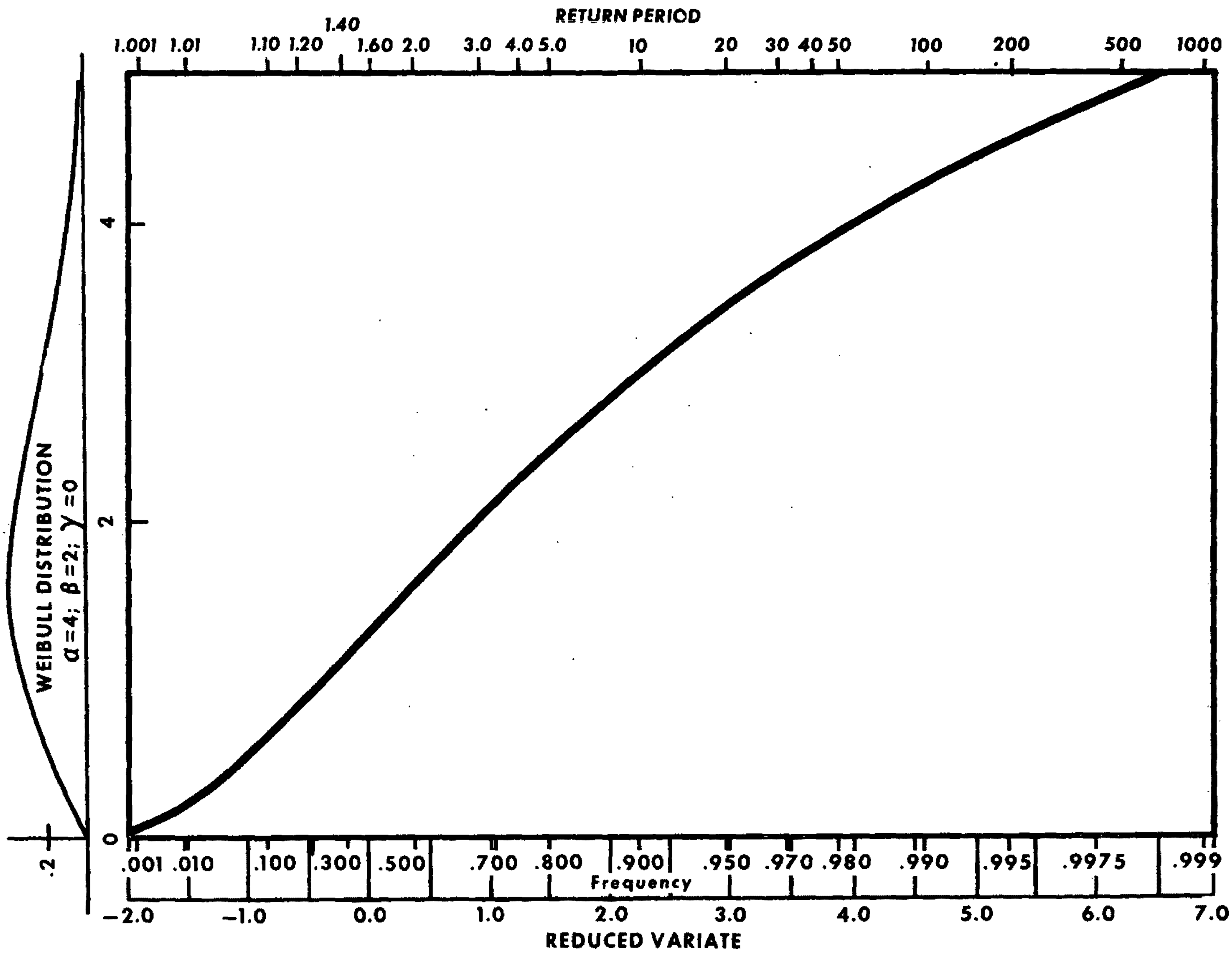

Figure 11. WEIBULL DISTRIBUTION ON GUMBEL PAPER

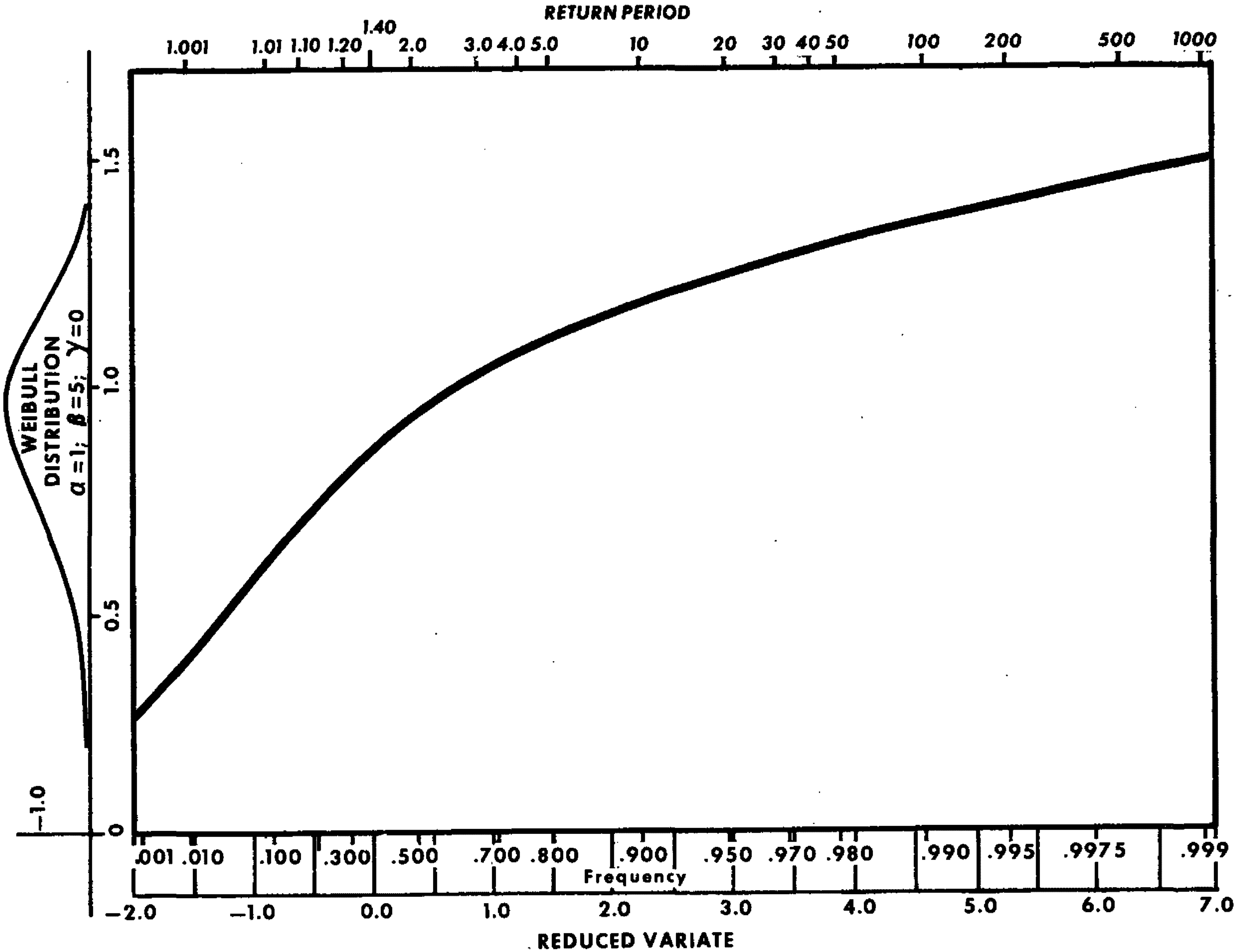

Figure 12. WEIBULL DISTRIBUTION ON GUMBEL PAPER

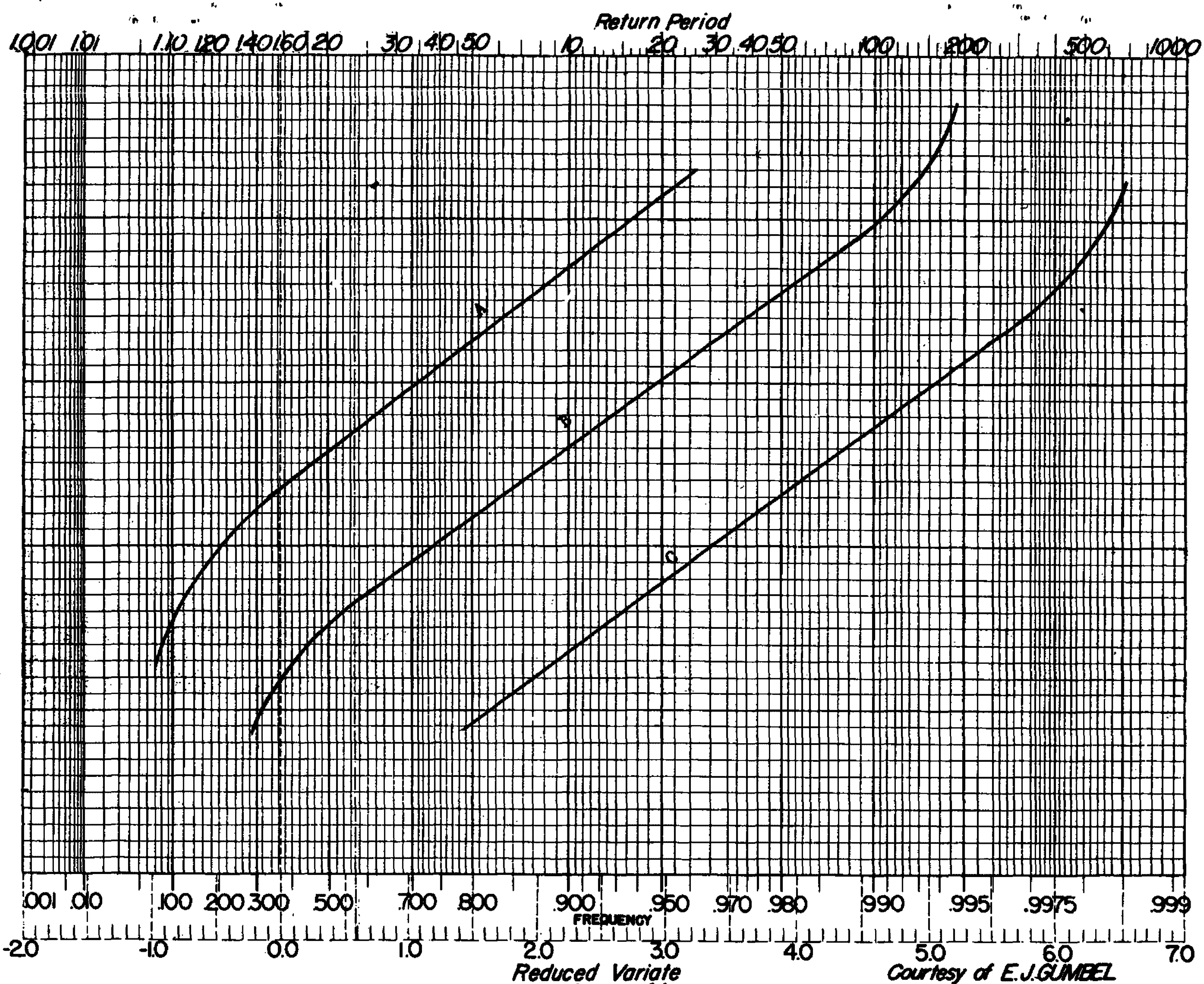

Figure 13

Gumbel plus outliers.

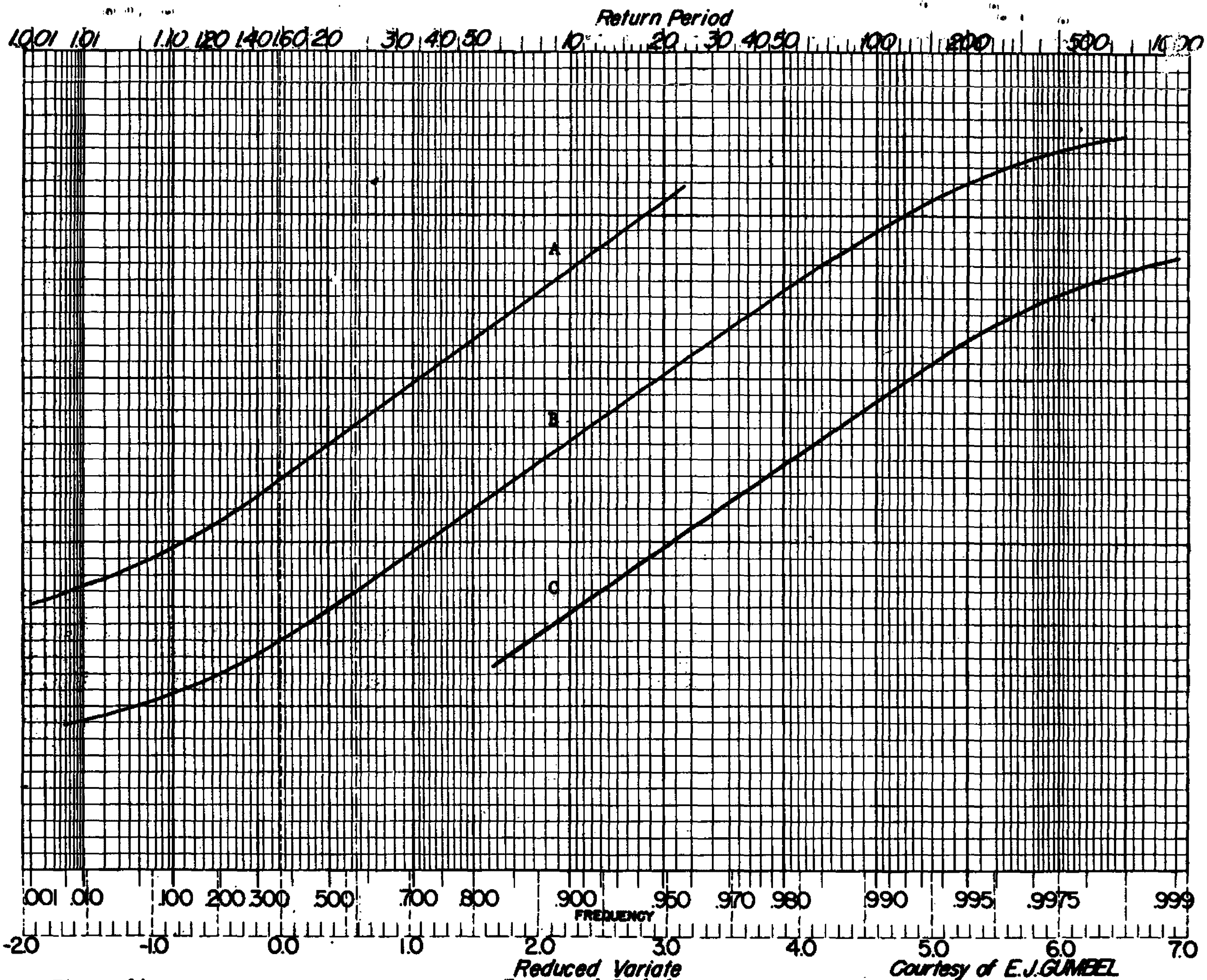

Figure 14

Truncated Gumbel.

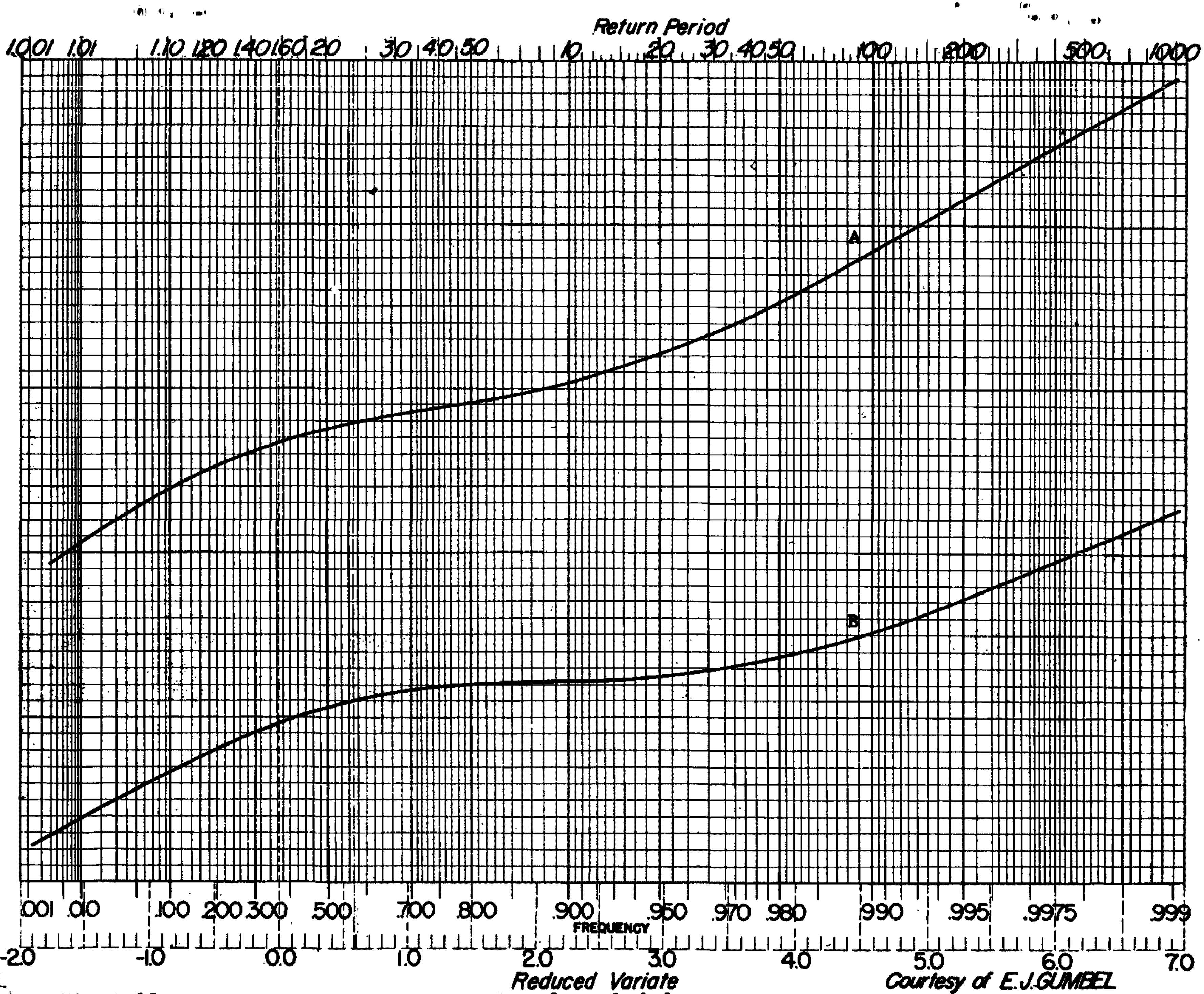

Figure 15
Sum of two Gumbels.

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