UNITED STATES DEPARTMENT OF COMMERCE • Luther H. Hodges, *Secretary*
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Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
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Inequalities for distribution functions

($F(x)$ denotes the c.d.f. of the random variable X and t denotes a positive constant; further m is always assumed to be finite and all expectations are assumed to exist.)

Inequality	Conditions
26.1.35 $Pr\{g(X)\geq t\}\leq E[g(X)]/t$	(i) $g(X)\geq 0$
26.1.36 $Pr\{X\geq t\}\leq m/t$ $F(t)\geq 1-\frac{m}{t}$	(i) $Pr\{X<0\}=0$ (ii) $E(X)=m$
26.1.37 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq 1/t^2$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{1}{t^2}$	(i) $E(X)=m$ * (ii) $E(X-m)^2=\sigma^2$
26.1.38 $Pr\{\lvert\overline{X}-\overline{m}\rvert\geq t\overline{\sigma}\}\leq\frac{1}{nt^2}$	(i) $E(X_i)=m_i$ (ii) $E(X_i-m_i)^2=\sigma_i^2$ (iii) $E([X_i-m_i][X_j-m_j])=0(i\neq j)$ (iv) $\overline{X}=\sum_{i=1}^{n}\frac{X_i}{n}$, $\overline{m}=\sum_{i=1}^{n}\frac{m_i}{n},\ \overline{\sigma}=\left[\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right]^{\frac{1}{2}}$
26.1.39 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq\frac{4}{9}\left\{\frac{1+\left(\frac{m-x_0}{\sigma}\right)^2}{\left(t-\left\lvert\frac{m-x_0}{\sigma}\right\rvert\right)^2}\right\}$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{4}{9}\left\{\frac{1+\left(\frac{m-x_0}{\sigma}\right)^2}{\left(t-\left\lvert\frac{m-x_0}{\sigma}\right\rvert\right)^2}\right\}$	(i) $E(X-m)^2=\sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0[6]
26.1.40 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq 4/9t^2$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{4}{9t^2}$	(i) $E(X-m)^2=\sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0[6] (iv) $m=x_0$
26.1.41 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq\frac{\mu_4-\sigma^4}{\mu_4+t^4\sigma^4-2t^2\sigma^4}$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{\mu_4-\sigma^4}{\mu_4+t^4\sigma^4-2t^2\sigma^4}$	(i) $E(X-m)^2=\sigma^2$ (ii) $E(X-m)^4=\mu_4$

[6] x_0 is such that $F'(x_0)>F'(x)$ for $x\neq x_0$.

26.2. Normal or Gaussian Probability Function

26.2.1 $$Z(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

26.2.2 $$P(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt=\int_{-\infty}^{x}Z(t)dt$$

26.2.3 $$Q(x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-t^2/2}dt=\int_{x}^{\infty}Z(t)dt$$

26.2.4 $$A(x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-t^2/2}dt=\int_{-x}^{x}Z(t)dt$$

26.2.5 $$P(x)+Q(x)=1$$

26.2.6 $$P(-x)=Q(x)$$

26.2.7 $$A(x)=2P(x)-1$$

Probability Integral with Mean m and Variance σ^2

A random variable X is said to be normally distributed with mean m and variance σ^2 if the probability that X is less than or equal to x is given by

26.2.8
$$Pr\{X\leq x\}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-m)^2}{2\sigma^2}}dt$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x-m)/\sigma}e^{-t^2/2}dt=P\left(\frac{x-m}{\sigma}\right).$$

The corresponding probability density function is

26.2.9
$$\frac{\partial}{\partial x}P\left(\frac{x-m}{\sigma}\right)=\frac{1}{\sigma}Z\left(\frac{x-m}{\sigma}\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}$$

and is symmetric around m, i.e.

$$Z\left(\frac{m+x}{\sigma}\right)=Z\left(\frac{m-x}{\sigma}\right).$$

The inflexion points of the probability density function are at $m\pm\sigma$.

Power Series ($x \geq 0$)

26.2.10 $$P(x)=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!2^n(2n+1)}$$

26.2.11
$$P(x)=\frac{1}{2}+Z(x)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1\cdot 3\cdot 5\ldots(2n+1)}$$

Asymptotic Expansions ($x>0$)

26.2.12

$$Q(x)=\frac{Z(x)}{x}\left\{1-\frac{1}{x^2}+\frac{1\cdot 3}{x^4}+\ldots+\frac{(-1)^n 1\cdot 3\ldots(2n-1)}{x^{2n}}\right\}+R_n$$

where

$$R_n=(-1)^{n+1}1\cdot 3\ldots(2n+1)\int_x^{\infty}\frac{Z(t)}{t^{2n+2}}\,dt$$

which is less in absolute value than the first neglected term.

26.2.13

$$Q(x)\sim\frac{Z(x)}{x}\left\{1-\frac{a_1}{x^2+2}+\frac{a_2}{(x^2+2)(x^2+4)}-\frac{a_3}{(x^2+2)(x^2+4)(x^2+6)}+\ldots\right\}$$

where $a_1=1$, $a_2=1$, $a_3=5$, $a_4=9$, $a_5=129$ and the general term is

$$a_n=c_0 1\cdot 3\ldots(2n-1)+2c_1 1\cdot 3\ldots(2n-3)+2^2c_2 1\cdot 3\ldots(2n-5)+\ldots+2^{n-1}c_{n-1}$$

and c_s is the coefficient of t^{n-s} in the expansion of $t(t-1)\ldots(t-n+1)$.

Continued Fraction Expansions

26.2.14

$$Q(x)=Z(x)\left\{\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\,\frac{4}{x+}\cdots\right\}\qquad(x>0)$$

26.2.15

$$Q(x)=\frac{1}{2}-Z(x)\left\{\frac{x}{1-}\,\frac{x^2}{3+}\,\frac{2x^2}{5-}\,\frac{3x^2}{7+}\,\frac{4x^2}{9-}\cdots\right\}\qquad(x\geq 0)$$

Polynomial and Rational Approximations[7] for $P(x)$ and $Z(x)$

$$0\leq x<\infty$$

26.2.16

$$P(x)=1-Z(x)(a_1t+a_2t^2+a_3t^3)+\epsilon(x),\qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|<1\times 10^{-5}$$

$p=.33267$ $\quad a_1=.43618\ 36$

$a_2=-.12016\ 76$

$a_3=.93729\ 80$

26.2.17

$$P(x)=1-Z(x)(b_1t+b_2t^2+b_3t^3+b_4t^4+b_5t^5)+\epsilon(x),\qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|<7.5\times 10^{-8}$$

$p=.23164\ 19$

$b_1=\ .31938\ 1530$ $\quad b_4=-1.82125\ 5978$

$b_2=-.35656\ 3782$ $\quad b_5=\ 1.33027\ 4429$

$b_3=\ 1.78147\ 7937$

26.2.18

$$P(x)=1-\frac{1}{2}(1+c_1x+c_2x^2+c_3x^3+c_4x^4)^{-4}+\epsilon(x)$$

$$|\epsilon(x)|<2.5\times 10^{-4}$$

$c_1=.196854$ $\quad c_3=.000344$

$c_2=.115194$ $\quad c_4=.019527$

26.2.19

$$P(x)=1-\frac{1}{2}(1+d_1x+d_2x^2+d_3x^3+d_4x^4+d_5x^5+d_6x^6)^{-16}+\epsilon(x)$$

$$|\epsilon(x)|<1.5\times 10^{-7}$$

$d_1=.04986\ 73470$ $\quad d_4=.00003\ 80036$

$d_2=.02114\ 10061$ $\quad d_5=.00004\ 88906$

$d_3=.00327\ 76263$ $\quad d_6=.00000\ 53830$

26.2.20 $$Z(x)=(a_0+a_2x^2+a_4x^4+a_6x^6)^{-1}+\epsilon(x)$$

$$|\epsilon(x)|<2.7\times 10^{-3}$$

$a_0=2.490895$ $\quad a_4=-.024393$

$a_2=1.466003$ $\quad a_6=\ .178257$

[7] Based on approximations in C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

26.2.21

$$Z(x)=(b_0+b_2x^2+b_4x^4+b_6x^6+b_8x^8+b_{10}x^{10})^{-1}+\epsilon(x)$$

$$|\epsilon(x)|<2.3\times10^{-4}$$

$b_0=2.50523\ 67$	$b_6=\ .13064\ 69$
$b_2=1.28312\ 04$	$b_8=-.02024\ 90$
$b_4=\ .22647\ 18$	$b_{10}=\ .00391\ 32$

Rational Approximations [7] for x_p where $Q(x_p)=p$

$$0<p\leq .5$$

26.2.22

$$x_p=t-\frac{a_0+a_1t}{1+b_1t+b_2t^2}+\epsilon(p), \qquad t=\sqrt{\ln\frac{1}{p^2}}$$

$$|\epsilon(p)|<3\times10^{-3}$$

$a_0=2.30753$	$b_1=.99229$
$a_1=\ .27061$	$b_2=.04481$

26.2.23

$$x_p=t-\frac{c_0+c_1t+c_2t^2}{1+d_1t+d_2t^2+d_3t^3}+\epsilon(p), \qquad t=\sqrt{\ln\frac{1}{p^2}}$$

$$|\epsilon(p)|<4.5\times10^{-4}$$

$c_0=2.515517$	$d_1=1.432788$
$c_1=\ .802853$	$d_2=\ .189269$
$c_2=\ .010328$	$d_3=\ .001308$

Bounds Useful as Approximations to the Normal Distribution Function

26.2.24

$$P(x)\leq\begin{cases} P_1(x)=\frac{1}{2}+\frac{1}{2}\,(1-e^{-2x^2/\pi})^{\frac{1}{2}} & (x>0)\\ P_2(x)=1-\frac{(4+x^2)^{\frac{1}{2}}-x}{2}\,(2\pi)^{-\frac{1}{2}}e^{-x^2/2} & \\ & (x>1.4)\end{cases}$$

26.2.25

$$P(x)\geq\begin{cases} P_3(x)=\frac{1}{2}+\frac{1}{2}\left(1-e^{-2x^2/\pi}-\frac{2(\pi-3)}{3\pi^2}\,x^4e^{-x^2/2}\right)^{\frac{1}{2}} & \\ & (x>0)\\ P_4(x)=1-\frac{1}{x}\,(2\pi)^{-\frac{1}{2}}e^{-x^2/2} & (x>2.2)\end{cases}$$

See **Figure 26.1** for error curves.

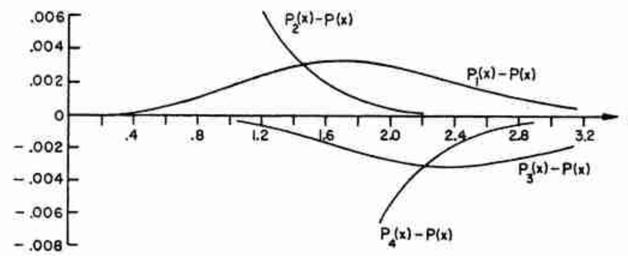

FIGURE 26.1. *Error curves for bounds on normal distribution.*

Derivatives of the Normal Probability Density Function

26.2.26 $$Z^{(m)}(x)=\frac{d^m}{dx^m}Z(x)$$

Differential Equation

26.2.27 $$Z^{(m+2)}(x)+xZ^{(m+1)}(x)+(m+1)Z^{(m)}(x)=0$$

Value at $x=0$

26.2.28

$$Z^{(m)}(0)=\begin{cases}\dfrac{(-1)^{m/2}m!}{\sqrt{2\pi}2^{m/2}\left(\dfrac{m}{2}\right)!} & \text{for } m=2r, r=0, 1, \ldots\\ 0 & \text{for odd } m>0\end{cases}$$